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DEFORMATION OF ELASTIC BODIES WITH THIN LIGAMENTS†

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A body with a hole in it has a thin ligament if the boundary of the hole approaches the outer surface of the body. The asymptotic form of the stress–deformation state of two- and three-dimensional bodies with ligaments is determined, using the width of the ligament as a small parameter. A boundary-layer effect arises near the ligament and can be described, in the two-dimensional case, by a system of ordinary differential equations which can be solved explicitly. The stress–deformation state turns out to depend closely both on the value characterizing the degree to which the ligament has narrowed, and on the overall geometric structure of the body. Analysis of the asymptotic formulae indicates that the collapse of a ligament cannot be a quasistatic process (the Griffith energy balance is destroyed). In the three-dimensional case, the boundary layer is described by an elliptic system of equations in the plane.

1. STATEMENT OF THE TWO-DIMENSIONAL PROBLEM

LET G_0 and G be regions in the plane, bounded by simple smooth closed contours Γ_0 and Γ which touch at the origin of coordinates O, and let $G_0 \subset G$. We reduce the characteristic size of the region G_0 to unity and define (dimensionless) Cartesian coordinates $x = (x_1, x_2)$ by taking the Ox_1 axis along the tangent to Γ_0, and the Ox_2 axis into G_0. Let $0 < \epsilon$ be a small parameter and let $G_\epsilon = \{x: (x_1, x_2 - \epsilon) \in G_0\}$, $\Gamma_\epsilon = \partial G_\epsilon$, $\Omega_\epsilon = G \backslash G_\epsilon$ (Fig. 1). In a small neighbourhood V of the point O it is assumed that the region Ω_ϵ is defined by the relation

$$-h_-(x_1) < x_2 < \epsilon + h_+(x_1) \tag{1.1}$$

$$h_\pm(x_1) = x_1^{2m}(a_\pm + O(x_1)), \quad x_1 \to 0 \tag{1.2}$$

In (1.1) and (1.2), $h_\pm$ are smooth functions, $h = h_+ + h_- > 0$, m is a natural number and $a = a_+ + a_- > 0$.

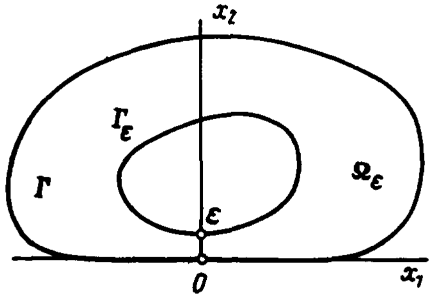

FIG. 1.

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The purpose of this paper is to investigate the asymptotic form as $\epsilon \to 0$ of the solution of the problem of the plane deformation of a body Ω_ϵ with a thin ligament

$$\mu \nabla_x \cdot \nabla_x u(\epsilon, x) + (\lambda + \mu) \nabla_x \nabla_x \cdot u(\epsilon, x) = 0, \quad x \in \Omega_\epsilon \tag{1.3}$$

$$\sigma^{(n)}(u; \epsilon, x) = p(x), \quad x \in \Gamma, \quad \sigma^{(n)}(u; \epsilon, x) = 0, \quad x \in \Gamma_\epsilon \tag{1.4}$$

Here $u = (u_1, u_2)$ is the displacement vector, $\sigma(u)$ is the stress tensor, $\sigma^{(n)} = \sigma n$, $n = (n_1, n_2)$ is the unit vector of the external normal, and $p \in C^\infty(\Gamma)$ is the applied load.

The problem of a thin ligament formed by a circular hole which approaches the boundary of a half-plane has been studied in [1–3], where results on the asymptotic behaviour were obtained from the exact solution of the whole problem. Below, we make a direct asymptotic analysis, from which the asymptotic form for non-canonical regions, in which it is difficult to construct explicit solutions, can also be determined. In addition, we study the dependence of the asymptotic form of stresses in the ligament on the extent to which it narrows [on the index m in (1.2)]. Problems concerning a thin ligament between two parallel cracks treat a similar theme and use similar methods of investigation [4, 5].

In the limit as $\epsilon \to 0$, the sides of the ligament touch, such that the doubly-connected region Ω becomes a simply connected region Ω_0, the boundary of which contains a singular point O, the top of two peaks. Outside the neighbourhood of the ligament, the asymptotic form of the solution of problem (1.3) and (1.4) can be described by solutions of limit problems in the region Ω_0. A boundary-layer effect arises in the ligament. This boundary layer is found in Sec. 2 by constructing the asymptotic form of the solution of problems in thin regions [6–9] and using the "rapid" variables

$$\xi = (\xi_1, \xi_2), \quad \xi_1 = \epsilon^{-\gamma} x_1, \quad \xi_2 = \epsilon^{-1} x_2, \quad \gamma = (2m)^{-1} \tag{1.5}$$

A similar procedure also yields asymptotic expansions of the limit problem near the singular point O (see Sec. 3; a proof of the resulting formal expansion is given in [10, 11]). In Sec. 4, the method of matched asymptotic expansions (see [12, 13] and elsewhere) is used to find the global asymptotic behaviour. Corollaries and generalizations of the formulae obtained can be found in Secs 5 and 6. Finally, the analogous three-dimensional problem is discussed in Sec. 7.

2. THE ASYMPTOTIC FORM OF THE SOLUTION OF THE PROBLEM IN A THIN LIGAMENT

After changing to variables (1.5), the Lamé operator $L(\nabla_x)$ of (1.3) can be written as follows:

$$L(\epsilon^{-\gamma}\partial_1, \epsilon^{-1}\partial_2) = \epsilon^{-2}\{M^{(0)}\partial_2^2 + \epsilon^\alpha (M^{(11)} + M^{(12)})\partial_1\partial_2 + \epsilon^{2\alpha} M^{(2)}\partial_1^2\} \tag{2.1}$$

$$M^{(0)} = \operatorname{diag}\{\mu, 2\mu + \lambda\}, \quad M^{(2)} = \operatorname{diag}\{2\mu + \lambda, \mu\}$$

$$M_{12}^{(11)} = M_{21}^{(12)} = \mu, \quad M_{21}^{(11)} = M_{12}^{(12)} = \lambda$$

$$M_{ii}^{(11)} = M_{ii}^{(12)} = 0, \quad i = 1, 2; \quad \alpha = 1 - \gamma, \quad \partial_i = \partial/\partial\xi_i$$

The operator in the brackets in (2.1) contains a small parameter in some of its higher derivatives. It is therefore natural to use the algorithm of [6–9]. We will first derive an expansion of the differential operator $B^\pm(x, \nabla_x)$ similar to (2.1) from boundary conditions (1.4). According to (1.1), the equation of the boundary in coordinates ξ has the form

$$\xi_2 = \pm H_\pm(\epsilon, \xi_1) \tag{2.2}$$

$$H_+(\epsilon, \xi_1) = 1 + \epsilon^{-1} h_+(\epsilon^\gamma \xi_1), \quad H_-(\epsilon, \xi_1) = \epsilon^{-1} h_-(\epsilon^\gamma \xi_1).$$

We should emphasize that the function $H_\pm$ is bounded in the zone $|\xi_1| < \text{const}$, where the asymptotic form of the solution is being investigated. The vectors of the external normal $n_\pm(\epsilon, \xi_1)$ to the curves (2.2) are given by the formulae

$$n_\pm(\epsilon,\xi_1)=N_\pm(\epsilon,\xi_1)^{-1}(-H'_\pm(\epsilon,\xi_1),\pm 1)$$
$$N_\pm(\epsilon,\xi_1)=[1+(H'_\pm(\epsilon,\xi_1))^2]^{1/2};\quad H'_\pm(\epsilon,\xi_1)=\partial_1 H_\pm(\epsilon,\xi_1).$$

Thus

$$N_\pm(\epsilon,\xi_1)B^\pm(\epsilon^\gamma\xi_1,\epsilon^{-\gamma}\partial_1,\epsilon^{-1}\partial_2)=\epsilon^{-1}\{\pm M^{(0)}\partial_2+\epsilon^\alpha(\pm M^{(11)}\partial_1-H'_\pm M^{(12)}\partial_2)-$$
$$-\epsilon^{2\alpha}H'_\pm M^{(2)}\partial_1\}. \tag{2.3}$$

We will seek the asymptotic representation of the solution in the form of the series

$$u(\epsilon,x)\sim\epsilon^\tau\sum_{j=0}^{\infty}\epsilon^{\alpha j}(u^j(\xi_1)+U^j(\xi_1,\xi_2)),\quad u^j=(v^j,w^j). \tag{2.4}$$

The index τ in (2.4) will be chosen below. We substitute (2.1), (2.3) and (2.4) into Eqs (1.3) and (1.4), restricted to a neighbourhood of the point O, and collect coefficients of like powers of ϵ (the calculation is simpler if the dependence of $H_\pm$ on ϵ is disregarded). As a result, we obtain a recurrence relation of ordinary differential equations (with respect to the variable $\xi_2\in[-H_-(\epsilon,\xi_1),H_+(\epsilon,\xi_1)]$ to find the functions U^j of (2.4); the vector-functions u^j are assumed to be arbitrary for the time being. The conditions for these problems to be solvable for U^k give a system of ordinary differential equations (with respect to ξ_1), which v^j and w^j must satisfy. Only a few terms of the series (2.4) will be needed. We will therefore give expressions for the vectors U^j only for the case where the components $v^1\equiv v$ and $w^0\equiv w$ are non-zero and the other functions v^j and w^j are equal to zero

$$U_1^0=U_2^0=0;\quad U_1^1=-\xi_2\partial_1 w,\ U_2^1=0$$
$$U_1^2=0,\quad U_2^2=\lambda(\lambda+2\mu)^{-1}(\tfrac{1}{2}\xi_2^2\partial_1^2 w-\xi_2\partial_1 v)$$
$$U_1^3=(\lambda+2\mu)^{-1}\{(3\lambda+4\mu)(\tfrac{1}{6}\xi_2^3\partial_1^3 w-\tfrac{1}{2}\xi_2^2\partial_1^2 v)+(\lambda+\mu)\xi_2\partial_1[-(H_+^2+H_-^2)\partial_1^2 w+$$
$$+2(H_+-H_-)\partial_1 v]\},\quad U_2^3=0 \tag{2.5}$$

The given solvability conditions arise when determining U_1^3 and U_2^4 and have the form

$$\partial_1\{-\tfrac{1}{6}(H_+^3+H_-^3)\partial_1^3 w+\tfrac{1}{4}(H_++H_-)\partial_1[(H_+^2+H_-^2)\partial_1^2 w]\}-$$
$$-\partial_1\{\tfrac{1}{2}(H_++H_-)\partial_1(H_+-H_-)\cdot\partial^1 v\}=F_2^0 \tag{2.6}$$
$$-\partial_1(H_++H_-)\partial_1 v+\tfrac{1}{2}\partial_1(H_+^2-H_-^2)\partial_1^2 w=F_1 \tag{2.7}$$

Here F_1 and F_2^0 are certain functions on the real axis $\mathbf{R}$, defined by the right-hand side p of the original problem.

The system (2.6), (2.7) is not formally self-conjugate. To reduce it to symmetric form, we apply the operator $\tfrac{1}{2}\partial_1(H_+-H_-)$ to the second equation and add the result to the first. The system becomes self-conjugate after (2.6) has been replaced by the equation

$$-\tfrac{1}{2}\partial_1^2(H_+^2-H_-^2)\partial_1 v+\tfrac{1}{3}\partial_1^2(H_+^3+H_-^3)\partial_1^2 w=F_2 \tag{2.8}$$

Here $F_2=F_2^0+\tfrac{1}{2}\partial_1(H_+-H_-)F_1$.

Thus, the limiting problem ($\epsilon=0$) corresponding to a stress–deformation state in a ligament is described by system (2.7), (2.8) in which, according to (1.2), (1.5) and (2.2), we must put

$$H_+(\xi_1)=1+a_+\xi_1^{2m},\quad H_-(\xi_1)=a_-\xi_1^{2m} \tag{2.9}$$

We further need the solutions $\psi=(v,w)$ of the uniform system (2.7), (2.8) on the straight line $\mathbf{R}$. Three of them are obvious

$$\psi^1(\xi_1)=(1,0),\quad \psi^2(\xi_1)=(0,1),\quad \psi^3(\xi_1)=(0,\xi_1) \tag{2.10}$$

Another three are obtained by integrating the system. To shorten the formulae, we introduce the notation

$$(R_1z)(\xi_1)=\int_0^{\xi_1} z(t)dt-\tfrac{1}{2}\int_0^{\infty}(z(t)-z(-t))dt,\quad (R_2z)(\xi_1)=\int_0^{\xi_1}(\xi_1-t)z(t)dt+$$
$$+\tfrac{1}{2}\int_0^{\infty} t(z(t)+z(-t))dt-\tfrac{1}{2}\xi_1\int_0^{\infty}(z(t)-z(-t))dt \tag{2.11}$$

Note that $\partial_1 R_1 z = z$ and $\partial_1^2 R_2 z = z$. The solutions mentioned have the form

$$\psi^{j+3}=(R_1\Psi_1^j,\ R_2\Psi_2^j),\quad j=1,2,3 \tag{2.12}$$

$$\Psi_1^1=-4H^{-4}(H_+^3+H_-^3),\quad \Psi_2^1=-6H^{-4}(H_+^2-H_-^2)$$
$$\Psi_1^2=6\xi_1H^{-4}(H_+^2-H_-^2),\quad \Psi_2^2=12\xi_1H^{-3}$$
$$\Psi_1^3=-6H^{-4}(H_+^2-H_-^2),\quad \Psi_2^3=-12H^{-3};\quad H=H_++H_- \tag{2.13}$$

We also introduce solutions ψ^{0q} of the uniform system (2.7), (2.8), in which the quantities $H_\pm^0(\xi_1)=a_\pm\xi_1^{2m}$ are used instead of $H_\pm$ of (2.9). In addition to the obvious solutions (2.10), this system is satisfied by the vector-functions ψ^{0j+3}, $j=1,2,3$, where

$$\psi^{0j+3}(\xi_1)=A_+^{-3}\xi_1^{\beta_j}(b_1^j,b_2^j\xi_1^{1-2m}) \tag{2.14}$$

$$\beta_1=1-2m,\quad \beta_2=2-4m,\quad \beta_3=1-4m,\quad A_\pm=a_+\pm a_-$$

$$b_1^1=-4(a_+^3+a_-^3)(1-2m)^{-1}A_+^{-1},\qquad b_2^1=-3(1-2m)^{-1}(1-4m)^{-1}A_-$$
$$b_1^2=3(1-2m)^{-1}A_-,\qquad b_2^2=2(1-2m)^{-1}(1-3m)^{-1}$$
$$b_1^3=-6(1-4m)^{-1}A_-,\qquad b_2^3=-6(1-3m)^{-1}(1-6m)^{-1}$$

By virtue of (2.11), (2.13) and (2.9), the solutions (2.12) can be expanded in series

$$\psi^{j+3}(\xi_1)=\psi^{0j+3}(\xi_1)\pm\sum_{k=1}^{3}c_{jk}\psi^k(\xi_1)+(1,|\xi_1|^{1-2m})\cdot O(|\xi_1|^{\beta_j-1}),\ \xi_1\to\pm\infty \tag{2.15}$$

$$c_{12}=c_{21}=c_{32}=c_{31}=0$$

$$c_{11}=\int_0^\infty \Psi_1^1(t)dt,\quad c_{13}=c_{31}=\int_0^\infty \Psi_1^3(t)dt=\int_0^\infty \Psi_2^1(t)dt$$

$$c_{22}=-\int_0^\infty t\Psi_2^2(t)dt,\quad c_{33}=\int_0^\infty \Psi_2^3(t)dt \tag{2.16}$$

It follows from (2.16) and (2.13) that the matrix $\mathbf{c}=\|c_{jk}\|$ comprising the coefficients of the series (2.15) is symmetrical and negative definite. This is true of any functions $H_\pm$ which increase sufficiently rapidly at infinity. We denote the matrix differential operator of system (2.7), (2.8) by $T(\xi_1,\partial_1)$. The scalar product $\psi^{3+j}\cdot T\psi^{3+k}$ is integrated by parts in the interval $(-R,R)$, the terms outside the integral are calculated using the asymptotic formulae (2.15), and taking the limit as $R\to\infty$ we obtain the representation

$$2c_{jk}=\lim_{R\to\infty}\int_{-R}^{R}\psi^{j+3}(\xi_1)\cdot T(\xi_1,\partial_1)\psi^{k+3}(\xi_1)d\xi_1-\Xi(\psi^{j+3},\psi^{k+3};(-R,R))=$$
$$=-\Xi(\psi^{j+3},\psi^{k+3};(-\infty,+\infty))$$
$$\Xi(\varphi,\psi;I)=\int_I(H_++H_-)\partial_1\varphi_1\partial_1\psi_1+\tfrac{1}{3}(H_+^3+H_-^3)\partial_1^2\varphi_2\partial_1^2\psi_2-\tfrac{1}{2}(H_+^2-H_-^2)(\partial_1^2\varphi_2\partial_1\psi_1+\partial_1\varphi_1\partial_1^2\psi_2)d\xi_1$$
$$\Xi(\varphi,\varphi;I)\geqslant(4+2\cdot 3^{1/2})^{-1}\int_I(H_++H_-)|\partial_1\varphi_1|^2+\tfrac{1}{3}(H_+^3+H_-^3)|\partial_1^2\varphi_2|^2d\xi_1$$

Thus, $\mathbf{c}$ is the Gram matrix, which possesses the properties mentioned.

3. ASYMPTOTIC FORM OF THE SOLUTION FOR A DEGENERATE LIGAMENT

We will consider the limit problem ($\epsilon=0$) (1.3), (1.4). Since the load is self-balanced, a solution of the problem exists which possesses finite elastic energy. We will find the asymptotic form of the solution near $(0, +0)$ and $(0, -0)$, the tops of the peaks formed by the contours Γ_0 and Γ. The procedure for constructing the asymptotic form is essentially the same as that for finding the coefficients of (2.4), with the initial variables x being used instead of the rapid variables ξ, and the functions $h_\pm$ from (1.2) taken as the functions $H_\pm$ in (2.2). We should emphasize that the equations of the boundary $x_2 = \pm h_\pm(x_1)$ no longer contain a small parameter, but the ligament is still thin, by virtue of relation (1.2).

We will first consider the case when $p = 0$ in the neighbourhood of the point O. Then the asymptotic form of the solution u^0 of the limit problem (1.3), (1.4) in the region Ω_0 has the form

$$u^0(x) = (k_1^\pm - k_3^\pm x_2,\ k_2^\pm + k_3^\pm x_1) + O(\exp(-\delta_0 |x_1|^{1-2m})), \quad x_1 \to \pm 0 \tag{3.1}$$

Here $k_j^\pm$ are certain constants, and $0<\delta_0$ is a small number. Since the solution u^0 is defined to within rigid displacements, it can be chosen so that

$$k_j^\pm = \pm k_j, \quad j = 1, 2, 3 \tag{3.2}$$

We further need the displacement fields Z^j in the region Ω_0 corresponding to concentrated effects at the tops of the peaks (the analogues of the longitudinal and shear forces, and the bending moment). More precisely, the vectors Z^j satisfy the homogeneous equations (1.3) in Ω_0 and (1.4) on $\partial\Omega_0 \backslash O$ and near the tops of the peaks they possess the asymptotic forms

$$Z^j(x) \sim \pm \mathbf{U}(1, x_2, \partial/\partial x_1)\Psi^{j+3}(x), \quad x_1 \to \pm 0 \tag{3.3}$$

Let us explain the notation used in (3.3). $\mathbf{U}$ denotes the matrix differential operator, defined using vectors with components (2.5)

$$\mathbf{U}(\epsilon, \xi_2, \partial/\partial\xi_1)(v(\xi_1),\ w(\xi_1)) = (\epsilon^\alpha v(\xi_1),\ w(\xi_1)) + \sum_{k=1}^{4} \epsilon^{k\alpha}(U_1^k(\xi),\ U_2^k(\xi)) \tag{3.4}$$

Note that in (3.4) $U_1^4 = 0$, but the explicit form of U_2^4 is not required [the possibility of constructing it was pointed out after (2.5)]. The vector-functions Ψ^{j+3} are defined as follows. In the neighbourhood of the point $x_1 = 0$ with a hole, they satisfy the uniform equations (2.7) and (2.8), with $h_\pm(x_1)$ instead of $H_\pm(\xi_1)$, and according to (1.2) they can be expanded in series

$$\Psi^{j+3}(x_1) = \psi^{0j+3}(x_1) + (1, |x_1|^{1-2m}) \cdot O(|x_1|^{\beta_j+1}), \quad x_1 \to \pm 0 \tag{3.5}$$

The proof of the existence of the required solutions follows the usual scheme: the asymptotic terms $Z^{j\pm}$ of (3.3) are multiplied by cutting-off functions $\chi_\pm(x_1)$, the discrepancies f^j, p^j of the resulting products in the uniform limit problem ($\epsilon = 0$) (1.3), (1.4) are then calculated and, finally, the "energy" solutions Z^{1j} of the problem in Ω_0 which compensate the discrepancies are added. We will explain why it is possible to find solutions Z^{1j} with finite energy. The procedure used to construct the formal asymptotic form described at the beginning of Sec. 2, gives terms of the series U^k with $k = 4, 5, \ldots$. If $(v, w) = \psi^{3+j}$, then according to (3.5), (2.14) and (2.5),

$$|U^k(x)| \leq c_1 |x_1|^{\beta_j + (k-1)(2m-1)} \leq c_2 |x_1|^{(k-3)(2m-1)-1} \tag{3.6}$$

Thus, replacing the operator $\mathbf{U}$ in (3.3) by (3.4) with a large number of terms of the series, the discrepancies can be made to decrease as $|x| \to 0$ as rapidly as desired, and this removes the need to consider admissible singularities at the tops of peaks. Since, from (3.6), the derivatives of the functions U_i^5 are quadratically summable, it is sufficient to take four terms of (3.4) when constructing the energy correction Z^{1j}. Thus, for the existence of the required vector Z^{1j} it remains only to verify that the loads f^j, p^j are self-balancing. We will denote by $\Omega(d)$ the region $\Omega_0 \backslash Q_d$ with "broken-off" peaks, where $0<d$ is a small number, and $Q_d = \{x\colon |x_1|<d,\ |x_2|<cd\}$ is a rectangle whose boundary intersects Ω_0 in the segments $I^\pm(d) = \{x\colon x_1 = \pm d,\ -h_-(\pm d)<x_2<h_+(\pm d)\}$. We put $X^1 = e^1$, $X^2 = e^2$ and $X^3(x) = (-x_2, x_1)$. We have

$$\int_{\Omega_0} f^j \cdot X^k dx + \int_{\partial\Omega_0} p^j \cdot X^k ds_x = \lim_{d\to 0} \sum_{\pm} \Big(- \int_{\Omega(d)} X^k \cdot L(x_\pm Z^{j\pm}) dx + \int_{\partial\Omega(d)\cap\partial\Omega_0} X^k \cdot B^\pm(x_\pm Z^{j\pm}) ds_x\Big) = \lim_{d\to 0} \sum_{\pm} \mp \int_{I(d)} \{\sigma_{11}(Z^{j\pm}) X_1^k + \sigma_{12}(Z^{j\pm}) X_2^k\} dx_2 \equiv \lim_{d\to 0} \sum_{\pm} \mp J_{j,k}^\pm(d) \tag{3.7}$$

By virtue of (2.5) we have the equations

$$\begin{aligned}
&\sigma_{11}(Z^{j\pm}) = \Lambda(\partial_1\Psi_1^{j+3} - x_2\partial_1^2\Psi_2^{j+3}) + (3\lambda+4\mu)({}^1/_6 x_2^3\partial_1^4\Psi_2^{j+3} - {}^1\!/_2\partial_1^3\Psi_1^{j+3}) + \\
&+ (\lambda+\mu)x_2\partial_1^2(-(h_+^2+h_-^2)\partial_1^2\Psi_2^{j+3} + 2(h_+-h_-)\partial_1\Psi_1^{j+3}) \\
&\sigma_{12}(Z^{j\pm}) = \Lambda({}^1\!/_2 x_2^2\partial_1^3\Psi_2^{j+3} - x_2\partial_1^2\Psi_1^{j+3}) + {}^1\!/_4\Lambda\partial_1(-(h_+^2+h_-^2)\partial_1^2\Psi_2^{j+3} + 2(h_+-h_-)\partial_1\Psi_1^{j+3}), \\
&\partial_1 = \partial/\partial x_1, \quad \Lambda = 4\mu(\lambda+\mu)(\lambda+2\mu)^{-1}
\end{aligned} \tag{3.8}$$

Now, evaluating the integrals, we have

$$\mp J_{j,k}^{\pm} = \Lambda\delta_{j,k} + O(1), \quad d\to 0$$

Thus, the limits (3.7) become zero, that is, the loads are self-balancing and energy corrections Z^{1j} exist. We note that the same calculations (applied to the Betti formula for vectors u^0 and Z^j) can be used to find the constants k_j from (3.1) and (3.2)

$$k_j = -\int_\Gamma Z^j \cdot p\, dx, \quad j = 1, 2, 3 \tag{3.9}$$

It turns out that if $p(0)\neq 0$, there is no solution of problem (1.3), (1.4) which possesses finite energy in Ω_0. The procedure used to find the solutions u^0 with the smallest possible singularities is the same as before: we construct a part of the asymptotic series and select an asymptotic correction. Only the principal singular term of the asymptotic form is then needed. Returning to the algorithm of Sec. 2, we arrive at a system of ordinary differential equations (2.6), (2.7) in which the $H_\pm$ are replaced by $a_\pm x_1^{2m}$ and $F_1 = -\Lambda^{-1}p_1(0)$, $F_2(x) = \Lambda^{-1}(-p_2(0) + p_1()2a_- m x_1^{2m-1}$. The particular solution of this system has the form

$$(v(x_1), w(x_1)) = p_1(0)(\mathrm{v}^1(x_1), \mathrm{w}^1(x_1)) + p_2(0)(\mathrm{v}^2(x_1), \mathrm{w}^2(x_1)) \tag{3.10}$$

$$\begin{aligned}
&\mathrm{v}^1(x_1) = \Lambda^{-1}A_+^{-4}[4(a_+^3 + a_-^3) + 6(2m+1)^{-1}a_-(a_+^2 - a_-^2)](2-2m)^{-1}x_1^{2-2m} \\
&\mathrm{w}^1(x_1) = \Lambda^{-1}A_+^{-4}[6(a_+^2 - a_-^2) + 12(2m+1)^{-1}a_-(a_+ + a_-)](3-4m)^{-1}(2-4m)^{-1}x_1^{3-4m} \\
&\mathrm{v}^2(x_1) = -\Lambda^{-1}A_+^{-3}3(a_+ - a_-)(3-4m)^{-1}x_1^{3-4m} \\
&\mathrm{w}^2(x_1) = -\Lambda^{-1}A_+^{-3}6(3-6m)^{-1}(4-6m)^{-1}x_1^{4-6m}
\end{aligned} \tag{3.11}$$

If $m = 1$, the multiplier $(2-2m)^{-1}x_1^{2-2m}$ in the first row of (3.11) is replaced by $\ln|x_1|$.
The functions (3.11) define the asymptotic form of the solution

$$u^0(x) \sim \mathrm{U}(1, x_2, \partial/\partial x_1)\{p_1(0)(\mathrm{v}^1(x_1), \mathrm{w}^1(x_1)) + p_2(0)(\mathrm{v}^2(x_1), \mathrm{w}^2(x_1))\} \tag{3.12}$$

If $p_2(0)\neq 0$, the only important term is the second one in the brackets, which describes the principal term of the asymptotic form. The lowest-order terms after it, which depend, for example, on $(\partial_1 p)(0)$, are of a higher order than the expression $p_1(0)(\mathrm{v}^1, \mathrm{w}^1)$. This expression predominates if $p_2(0) = 0$.

4. THE ASYMPTOTIC FORM OF THE SOLUTION

We will assume first that $p = 0$ in the neighbourhood of the point O. For the principal approximation to the solution of the problem (1.3), (1.4), it is natural to take the solution u^0 of the limiting problem (1.3), (1.4) in Ω_0. This solution satisfies the boundary condition (1.4) on Γ, and outside the neighbourhood of the ligament it leaves a small error $O(\epsilon)$ in the condition on the contour Γ_ϵ. However, it is unsuitable as an approximation to the solution $u(\epsilon, x)$ near the point O, because it can have different limits as $x \to \pm 0$ [compare this with (3.1) and (3.2)]. Thus, using the method of matched asymptotic expansions, we select a different representation of the vector $u(\epsilon, x)$ on the ligament. For small x we shall therefore look for the principal term of the asymptotic form in the form

$$u(\epsilon, x) \sim \mathrm{U}\left(\epsilon, \xi_2, \frac{\partial}{\partial \xi_1}\right)\sum_{j=1}^{3} A_j(\epsilon)\psi^{j+3}(\xi_1) \tag{4.1}$$

Here ψ^{j+3} are the solutions of (2.12) described in Sec. 2, U is the operator of (3.4), and $A_j(\epsilon)$ are quantities to be determined. Using (2.15), we separate the non-decaying terms in the asymptotic form on the right-hand side of (4.1) as $\xi_1 \to \pm\infty$. Returning to x coordinates, according to (1.5), we find that these terms are equal to

$$\pm A_1(\epsilon)[\epsilon^\alpha c_{11}(1,0)+c_{13}(-\epsilon^\alpha \xi_2,\xi_1)] \pm A_2(\epsilon)c_{22}(0,1) \pm A_3(\epsilon)[\epsilon^\alpha c_{31}(1,0)+$$
$$+c_{33}(-\epsilon^\alpha\xi_2,\xi_1)] = \pm\epsilon^\alpha[c_{11}A_1(\epsilon)+c_{31}A_3(\epsilon)](1,0) \pm c_{22}A_2(\epsilon)(0,1) \pm$$
$$\pm\epsilon^{-\gamma}[c_{13}A_1(\epsilon)+c_{33}A_3(\epsilon)](-x_2,x_1)$$

Comparing the last expression with the asymptotic forms (3.1) and solving the system of algebraic equations using (3.2), we find

$$\begin{aligned} &A_2(\epsilon)=A_2^\circ=k_2c_{22}^{-1}\\ &A_p(\epsilon)=\epsilon^{-\alpha}A_p^\circ+O(\epsilon^\gamma) \quad (p=1,3)\\ &A_1^\circ=k_1c_{33}d^{-1},\ \ A_3^\circ=-k_1c_{13}d^{-1},\ \ d=c_{11}c_{33}-c_{13}^2\end{aligned} \tag{4.2}$$

Thus, the principal term of the asymptotic form of the solution $u(\epsilon,x)$ on the ligament must be found using (4.1), with $A_j(\epsilon)$, $j=1, 2, 3$ replaced by $\epsilon^{-\alpha}A_1^0$, A_2^0, $\epsilon^{-\alpha}A_3^0$ from (4.2). [Comparing (2.4) and (4.1), we see that $\tau=-\alpha$.] We now construct the next term $\epsilon^p u^p$ of the outer expansion. The choice of the power p of the small parameter is governed by two factors: by combination with the lowest-order terms (as $\xi_1\to\pm\infty$) of the expansion of the sum in (4.1) and by compensation of the error $O(\xi)$ left by the vector u^0 in the boundary condition (1.4) on Γ_ϵ. According to (2.15), (2.14) and (1.5), the lowest-order terms have the form

$$\begin{aligned}&U(\epsilon,\xi_2,\partial/\partial\xi_1)(A_1(\epsilon)\psi^{04}(\xi_1)+A_2(\epsilon)\psi^{05}(\xi_1)+A_3(\epsilon)\psi^{06}(\xi_1))+\ldots=\\ &=U(1,x_2,\partial/\partial x_1)\{\epsilon^{-\alpha}A_1^\circ\epsilon^{2-2\gamma}\psi^{04}(x_1)+A_2^\circ\epsilon^{3-3\gamma}\psi^{05}(x_1)+\\ &+\epsilon^{-\alpha}A_3^\circ\epsilon^{3-2\gamma}\psi^{06}(x_1)\}+\ldots\end{aligned} \tag{4.3}$$

The dots here denote unimportant terms. It is clear from (4.3) that the first term in brackets has the lowest index. This index $p\equiv\alpha=1-\gamma$ is less than one, and so the second term of the outer expansion is found by the matching procedure. Recalling the expansions (3.3) of special solutions Z^j, we conclude that

$$\epsilon^{1-\gamma}u^{1-\gamma}(x)=\epsilon^{1-\gamma}A_1^\circ Z^1(x) \tag{4.4}$$

When constructing the next terms of the asymptotic form, it is necessary to allow for discrepancies in the boundary condition on Γ and apply the matching procedure to the lowest terms of the series of special solutions. We shall merely point out the principal asymptotic correction (4.4) far away from the ligament and turn to the case $p(0)\neq 0$. Suppose first that $p_2\neq 0$; in (3.12), we change to rapid variables (1.5)

$$u^0(\epsilon,x)\sim\epsilon^{-4\alpha+'}\,U(\epsilon,\xi_2,\partial/\partial\xi_1)(v^2(\xi_1),w^2(\xi_1))$$

Thus, for the matching procedure, we need to construct a solution (v^2, w^2) of (2.7) and (2.8) with right-hand sides $F_1=0$, $F_2=-\Lambda^{-1}p_2(0)$, which has as its asymptotic form as $\xi_1\to\pm\infty$ the quantity $[v^2(\xi_1), w^2(\xi_1)]$. The answer is written using the operators R_i from (2.12)

$$\begin{aligned}&(v^2,w^2)=p_2(0)\Lambda^{-1}(R_1V^2,R_2W^2)\\ &V^2(\xi_1)=H(\xi_1)^{-4}[4c_1(H_+(\xi_1)^3+H_-(\xi_1)^3)-6(\tfrac{1}{2}\xi_1^2-c_2)(H_+(\xi_1)^2-H_-(\xi_1)^2)]\\ &W^2(\xi_1)=H(\xi_1)^{-4}[6c_1(H_+(\xi_1)^3+H_-(\xi_1)^3)-12(\tfrac{1}{2}\xi_1^2-c_2)(H_+(\xi_1)^2-H_-(\xi_1)^2)]\end{aligned} \tag{4.5}$$

$$\int_0^\infty V^2(\xi_1)d\xi_1=\int_0^\infty W^2(\xi_1)d\xi_1=0 \tag{4.6}$$

The functions $H_\pm$ in (4.5) are given by (2.9), and the constants c_1 and c_2 are found uniquely from (4.6). The principal term of the asymptotic form in the case $p_2(0) = 0$, $p_1 \neq 0$ has the form

$$u(\epsilon, x) \sim \epsilon^{-3\alpha+1} \mathbf{U}(\epsilon, \xi_2, \partial/\partial\xi_1)(\mathbf{v}^1(\xi_1), \mathbf{w}^1(\xi_1)) \tag{4.7}$$

We note that representations (4.4) and (4.7) agree with (2.4) for $\tau = -4\alpha + 1$ and $\tau = 3\alpha + 1$, respectively. If $m > 1$

$$(v^1, w^1) = p_1(0)\Lambda^{-1}(R_1 V^1, R_2 W^1)$$

$$V^1(\xi_1) = H(\xi_1)^{-4}[4\xi_1(H_+(\xi_1)^3 + H_-(\xi_1)^3) + 6(H_+^2(\xi_1) - H_-(\xi_1)^2)(\int_0^{\xi_1} H_-(t)dt + c_3\xi_1)]$$

$$W^1(\xi_1) = H(\xi_1)^{-4}[6\xi_1(H_+(\xi_1)^2 - H_-(\xi_1)^2) + 12(H_+(\xi_1) + H_-(\xi_1))(\int_0^{\xi_1} H_-(t)dt + c_3\xi_1)] ;$$

$$\int_0^\infty \xi_1 W^1(\xi_1)d\xi_1 = 0 \tag{4.8}$$

We emphasize that the last equation serves to find the constant c_3. Formulae (4.8) still hold when $m = 1$, but the odd functions V^1 and W^1 possess the asymptotic form

$$V^1(\xi_1) \sim 2A_+^{-3}(2a_+^2 - a_+a_- + a_-^2)\xi_1^{-1} \equiv K\xi_1^{-1}$$
$$W^1(\xi_1) \sim 2A_+^{-3}(3a_+ - a_-)\xi_1^{-3}$$

Relation (4.7) for w^1 makes sense, whereas the action of the operator R_1 on V^1 is not defined. It is therefore necessary to change the expression for the function v^1. We will put

$$v^1(\xi_1) = p_1(0)\Lambda^{-1}(\int_0^{\xi_1} V^1(\xi_1)d\xi_1 + c(\epsilon))$$

and the quantity $c(\epsilon)$ is chosen so that

$$v^1(\xi_1) = K\ln(\epsilon^{1/2}|\xi_1|) + O(|\xi_1|^{-1}), \quad |\xi_1| \to \infty \tag{4.9}$$

Not that, according to (1.5), changing from coordinates ξ to x eliminates $\ln\epsilon$ from (4.9). This circumstance allows the expansions (4.6) and (3.12) to be matched.

5. DISCUSSION

1. *Korn's weighted inequality*

In proving the solvability of the limit problem in Ω_0 and the estimate of the solution of the problem in Ω_ϵ, a special modification of Korn's inequality must be used. With the aid of the methods described in [14, 15] we obtain the following. Suppose that the vector $u \in W_2^1(\Omega_\epsilon)$ is subject to conditions which eliminate arbitrariness in the choice of the rigid displacement

$$\int_\Gamma (x_1 u_2(\epsilon, x) - x_2 u_1(\epsilon, x))\, ds_x = 0, \qquad \int_\Gamma u_i(\epsilon, x) ds_x = 0, \quad i = 1, 2$$

Then

$$E(u; \Omega_\epsilon) \geqslant c \int_{\Omega_\epsilon} \{(d+\epsilon)^{-2}|u_1|^2 + (d+\epsilon)^{-4}|u_2|^2 + \left|\frac{\partial u_1}{\partial x_1}\right|^2 + \left|\frac{\partial u_2}{\partial x_2}\right|^2 + (d+\epsilon)^{-2}\left|\frac{\partial u_1}{\partial x_2}\right|^2 +$$
$$+ (d+\epsilon)^{-2}\left|\frac{\partial u_2}{\partial x_1}\right|^2\} dx \tag{5.1}$$

in which E is a functional of the elastic energy and $d(x) = |x|^{2m}$. The constant c is independent of both u and $\epsilon \in (0, \epsilon_0]$, and inequality (5.1) remains true even when $\epsilon = 0$.

2. *Justification of the asymptotic form*

In previous sections, we constructed inner and outer expansions $\mathbf{u}^{\text{int}}(\epsilon, \xi)$ and $\mathbf{u}^{\text{ext}}(\epsilon, x)$ for the solution u. In view of various features of the construction of the global asymptotic approximation (compare this with [13, 16]), we will describe it in detail. Let χ be a smooth cutting-off function in Ω_ϵ; it is equal to zero outside the set $V \cap \overline{\Omega}_\epsilon$, and on the ligament $\chi(x) = \chi_0(x_1)$, where $x_0(x_1) = 1$ for $|x_1| < \rho_2$ and $\rho_0 > 0$. Suppose also that $\mathbf{U}^{\text{ext}}(\epsilon, x_1, h(x_1)^{-1}[x_2 + h_-(x_1)])$ represents the outer approximation $\mathbf{u}^{\text{ext}}(\epsilon, x)$ on the ligament in special variables. In addition, we assume that the vector function $\mathbf{u}^{\text{ext}}$ is continued and remains smooth beyond the set $\Omega_0 \backslash O$. The global approximation mentioned above has the form

$$(1 - \chi(x))\mathbf{u}^{\text{ext}}(\epsilon, x) + \chi(x)(1 - \chi_0(\epsilon^{-\gamma}x_1)) + \chi(x)(1 - \chi_0(\epsilon^{-\gamma}x_1))\mathbf{U}^{\text{ext}}(\epsilon, x_1, [\epsilon + h(x_1)]^{-1} \times$$
$$\times [x_2 + h_-(x_1)]) + \chi(x)\mathbf{u}^{\text{int}}(\epsilon, \epsilon^{-\gamma}x_1, \epsilon^{-1}x_2) - \chi(x)(1 - \chi_0(\epsilon^{-\gamma}x_1))\mathbf{U}^{as}(\epsilon, x) \tag{5.2}$$

Here $\mathbf{U}^{as}(\epsilon, x)$ denotes the general terms of the expansions of the vector functions $\mathbf{u}^{\text{ext}}$ and $\mathbf{u}^{\text{int}}$ as $x \to 0$ and $x_1 \to \infty$. This quantity is allowed for twice in (5.1), in both the second and the third terms, but this can be remedied by subtracting it. Substituting (5.2) into (1.3), (1.4), calculating the corresponding discrepancy and applying inequality (5.1), we obtain an energy estimate for the difference between the true solution u of problem (1.3), (1.4) and the asymptotic solution (5.2). We note that, on the basis of local estimates of the solutions of elliptic boundary problems [17], pointwise closeness of the above solutions can be established.

3. *Stress concentration on the ligament*

We will first consider the situation when $p(0) = 0$. According to Secs 3 and 4, far away from the ligament the asymptotic form of the solution has the form

$$u(\epsilon, x) \sim u^0(x) + \epsilon^{1-\gamma}u^{1-\gamma}(x) + \dots$$

Thus, outside the neighbourhood of the point O, the stresses $\sigma(u; \epsilon, x)$ are bounded. Owing to the singularity of the vector (4.4) at zero [compare this with (3.3), (3.5)], these stresses increase as $x \to 0$. Thus, the stress concentration observed on the ligament is defined by the inner expansion (4.1) (boundary-layer type solution). Starting from (4.1), (4.2), (2.12), (2.13) and (2.5) we have

$$\sigma_{11}(u; \epsilon, x) \sim \epsilon^{-\gamma}\Lambda \sum_{p=1,3} A_p^\circ(\Psi_1^p(\xi_1) - \xi_2\Psi_2^p(\xi_1)) + \dots$$
$$\sigma_{12}(u; \epsilon, x) \sim \epsilon^{-\gamma+\alpha}\Lambda \sum_{p=1,3} A_p^\circ[\tfrac{1}{2}\xi_2^2\partial_1\Psi_2^p(\xi_1) - \xi_2\partial_1\Psi_1^p(\xi_1) + D(\Psi^p; \xi_1)] + \dots$$
$$\sigma_{22}(u; \epsilon, x) \sim \epsilon^{-\gamma+\alpha}\lambda(\lambda + 2\mu)^{-1} \sum_{p=1,3} A_p^\circ[(3\lambda + 4\mu)(\tfrac{1}{6}\xi_2^3\partial_1^2\Psi_2^p(\xi_1) - \tfrac{1}{2}\xi_2^2\partial_1^2\Psi_1^p(\xi_1)) +$$
$$+ (\lambda + \mu)\partial_1 D(\Psi^p; \xi_1)] + \dots$$
$$D(\Psi; \xi_1) = \partial_1\{-(H_+(\xi_1)^2 + H_-(\xi_1)^2)\Psi_2(\xi_1) + 2(H_+(\xi_1) - H_-(\xi_1))\Psi_1(\xi_1)\}, \ \partial_1 = \partial/\partial\xi_1$$

It is clear that the component $\sigma_{11}(u)$ is the largest; in the main, it is a linear function of the variable x_2 and has order $\epsilon^{-\gamma}$.

If $p(0) \neq 0$, the calculations are carried out using formulae (4.4), (4.5) and (4.7), (4.8). Again, the highest-order term is the stress $\sigma_{11}(u)$, where

$$\sigma_{11}(u; \epsilon, x) \sim \epsilon^{-1+\gamma}p_1(0)(V^1(\xi_1) - \xi_2 W^1(\xi_1)) + \epsilon^{-2+2\gamma}p_2(0)(V^1(\xi_1) - \xi_2 W^2(\xi_1)) + \dots$$

In other words, when $p_1(0) \neq 0$, $p_2(0) = 0$, the stress $\sigma_{11}(u)$ on the ligament is of $O(\epsilon^{-1+\gamma})$. If $p_2(0) \neq 0$, $\sigma_{11}(u)$ has order $\epsilon^{-2+2\gamma}$.

4. *Modifications of the geometric shapes*

The region illustrated in Fig. 1 remains connected. Another possibility that could be considered is that, when $\epsilon = 0$, the region Ω_ϵ could split into two sets Ω_0^+ and Ω_0^- (Fig. 2). If the load p applied to each of the contours $\partial\Omega_0^\pm$ is self-balanced, the construction algorithm for the asymptotic form of the solution is simplified and the boundary layer can be found without the use of the vector function (3.3). The point is that there is an energy solution $u^{0\pm}$ of the limit problem in the region $\Omega_0^\pm$ which can be found apart from the rigid displacements and we can therefore assume that $u^{0\pm}(x) = o(\exp(-\delta_0|x_1|^{1-2m}))$as $x_1 \to \pm 0$. This means that we need only allow for discrepancies which arise due to regular perturbation of the boundary. But if the principal vector $(\mathbf{F}_1^\pm, \mathbf{F}_2^\pm)$ and principal moment $\mathbf{F}_3^\pm$ of a load applied to $\partial\Omega_0^\pm$ are non-zero, there is no energy solution $u^{0\pm}$. Taking into account obvious relations $(\mathbf{F}_j^\pm = \pm\mathbf{F}_j)$ and repeating the calculations of Sec. 3, we can see that $u^{0\pm}$ of the limit problem exists which possesses the asymptotic form

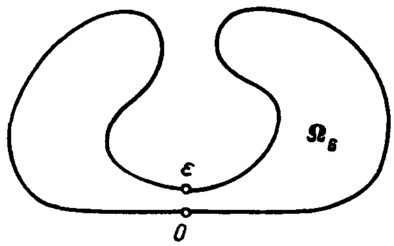

FIG. 2.

$$u^{0\pm}(x) = \pm \sum_{j=1}^{3} \mathbf{F}_j \mathbf{U}(1, x_2, \partial/\partial x_1)\Psi^{j+3}(x), \quad x_1 \to \pm 0 \tag{5.3}$$

Performing the matching, we look for the inner expansion in the form (4.1), where $A_j(\epsilon)$ is given as follows:

$$A_1(\epsilon) = \epsilon^{-2\alpha}\mathbf{F}_1, \quad A_2(\epsilon) = \mathbf{F}_2, \quad A_3(\epsilon) = \epsilon^{-2\alpha-1}\mathbf{F}_3 \tag{5.4}$$

We should emphasize that the sum over $k = 1, 2, 3$ of the expansions (2.15) of the solutions ψ^{j+3} does not affect the principal terms of the asymptotic form of the stresses $\sigma_{jk}(u;\ \epsilon,\ x)$, since the solutions (2.10) correspond to rigid displacements which constitute the arbitrariness in the choice of the solutions of the limit problem in $\Omega_0^{\pm}$. The stresses on the ligament are calculated with (4.1), (5.4). We should merely point out that when $\mathbf{F}_3 \neq 0$, $\sigma_{11}(u)$ has order ϵ^{-2}, and when $\mathbf{F}_3 = 0$ and $\mathbf{F}_2 \neq 0$, this correction is equal to $\epsilon^{-2+\gamma}$. Finally, if $\mathbf{F}_3 = \mathbf{F}_2 = 0$ and $\mathbf{F}_1 \neq 0$, then $\sigma_{11}(u;\epsilon,x) = O(\epsilon^{-1})$. In conclusion, we note that the procedure given here can be extended to the case where two bodies are connected by several ligaments (such as strips with a round hole).

5. *Fracture of a ligament*

We will now consider the region shown in Fig. 1, and suppose that $p = 0$ near the point O. Using (4.4), and the fact that the global approximation of the solution on $\Gamma\backslash V$ is the same on the whole as the sum $u^0(x) + \epsilon^{1-\gamma}A_1^\circ Z^1(x)$, we calculate the asymptotic expression of the potential energy of deformation of the body Ω_ϵ

$$\Pi(u;\ \Omega_\epsilon) = E(u;\ \Omega_\epsilon) - \int_\Gamma u \cdot p\,ds_x = -\tfrac{1}{2}\int_\Gamma u \cdot p\,ds_x = -\tfrac{1}{2}\int_\Gamma u^0 \cdot p\,ds_x - \tfrac{1}{2}\epsilon^{1-\gamma}A_1^\circ \int_\Gamma Z^1 \cdot p\,ds_x + O(\epsilon) =$$
$$= \Pi(u^0;\ \Omega_0) + \epsilon^{1-\gamma}\mathbf{c}_{33}k_1^2\mathbf{d}^{-1} + O(\epsilon) \tag{5.5}$$

We note that $c_{33}<0$, $\mathbf{d}>0$, that is, the second term in (5.5) is negative. Let Ω'_ϵ be a region in which the ligament has been torn apart (so that the arc joining points on opposite sides of the ligament is supplementing the boundary $\partial\Omega_\epsilon$). As in 4, in the asymptotic expansion of the solution $u'(\epsilon, x)$ of (1.3), (1.4) in Ω_ϵ, the boundary layer (4.1) and correction (4.4) to the outer expansion disappear, so that on $\Gamma\backslash V$ $u'(\epsilon,x) = u^0(x) + O(\epsilon)$. Thus $\Pi(u';\ \Omega'_\epsilon) = \Pi(u^0;\ \Omega_0) + O(\epsilon)$. This means that the increment of potential energy of deformation is $\epsilon^{1-\gamma}c_{33}k_1^2\mathbf{d}^{-1} + O(\epsilon)$. The increment of surface energy is $O(\epsilon)$ and, therefore, the energy balance is destroyed in the case of small ϵ. Thus, quasistatic fracture of a ligament is impossible within the framework of the Griffith hypothesis.

6. DIRICHLET'S PROBLEM FOR A BIHARMONIC EQUATION

In the region Ω_ϵ described in Sec. 1, we consider the boundary-value problem

$$\Delta^2 w(\epsilon, x) = 0, \quad x \in \Omega_\epsilon \tag{6.1}$$

$$w(\epsilon, x) = \varphi^-(x), \quad \partial_n w(\epsilon, x) = \psi^-(x), \quad x \in \Gamma \tag{6.2}$$

$$w(\epsilon, x) = \varphi^+(x), \quad \partial_n w(\epsilon, x) = \psi^+(x), \quad x \in \Gamma_\epsilon \tag{6.3}$$

This problem corresponds, for instance, to the bending of a plate with a rigidly fixed edge: in that case Ω_ϵ is a surface in the middle of the plate, and inhomogeneity on the right-hand side of (6.1) is eliminated by partial solution of a biharmonic equation in the plane. The Dirichlet conditions simplify the algorithm for constructing the asymptotic form; a boundary-layer type solution is calculated, whatever the solution of the limit problem in Ω_0 (see [16]). In fact, on the ligament, the asymptotic series is sought in the form

$$w(\epsilon, x) = W_0(t) + \sum_{k=1}^{\infty} \epsilon^{2\alpha k} W_k(x_1, t)$$
$$t = [\epsilon + h(x_1)]^{-1}(x_2 - h_-(x_1)) \tag{6.4}$$

Substituting (6.4) into (6.1), we find that W_0 is a cubic polynomial in the variable i. The coefficients of this polynomial are found for boundary conditions (6.2) and (6.3)

$$W_0(t) = \varphi^-(0) + [\varphi^+(0) - \varphi^-(0)](3t^2 - 2t^3) \tag{6.5}$$

All the terms of the series (6.4) can be found, as in [13, 16], by writing the biharmonic operator and the derivative ∂_n using the variables x_1, t and expanding the smooth functions $\varphi^\pm$, $\psi^\pm$ in a Taylor series.

Another possible way of interpreting the system (6.1)–(6.3) is as a problem for finding the Airy function. At first glance it would appear that the asymptotic form of the stress $\sigma(u; \epsilon, x)$ on the ligament can be found explicitly using (6.5), but this approach gives rise to complications, owing to the multiconnectedness of the region Ω.

We recall some of the properties of the Airy function (see [18] and elsewhere). It is defined to within a linear function and the arbitrariness can be removed, for instance, by the condition $\varphi^- = \psi^- = 0$. Let w be a solution of problem (6.1)–(6.3), for which the right-hand sides φ^+ and ψ^+ are calculated with respect to the vector $p = (p_1, p_2)$ from (1.4), using the formulae

$$\varphi^+(x(s)) = 1 + x_1(s) + x_2(s) + \int_0^s x_2 p_1 - x_1 p_2 d\tau$$
$$\psi^+(x(s)) = n_1(s)(1 - \int_0^s p_2(\tau)d\tau) + n_2(s)(1 + \int_0^s p_1(\tau)d\tau) \tag{6.6}$$

Here S is the length of the arc on Γ and $x(s)$ is the corresponding point; $x(0) = 0$. The functions (6.6) are smooth, because the load is self-balancing. Then let w_i be the solutions of the same problem with the following right-hand sides

$$\varphi_i^- = 0, \quad \varphi_0^+ = 1, \quad \varphi_k^+(x) = x_k$$
$$\psi_i^+ = 0, \quad \psi_0^- = 0, \quad \psi_k^-(x) = n_k(x), \quad k = 1, 2, \quad i = 0, 1, 2 \tag{6.7}$$

The Airy function obeys the relations

$$F = w + A_0 w_0 + A_1 w_1 + A_2 w_2 \tag{6.8}$$

$$F_{,22} = \sigma_{11}(u), \quad F_{,11} = \sigma_{22}(u), \quad F_{,12} = -\sigma_{12}(u) \tag{6.9}$$

The constants A_i are found from the condition that the displacement vector is uniquely defined by (6.9). It has been verified [19] that this condition can be written as follows:

$$\langle F, w_i \rangle = 0, \quad i = 0, 1, 2$$
$$\langle F, G \rangle = \sum_{j,k=1}^{2} \int_\Omega \left(\frac{\partial^2 F}{\partial x_j \partial x_k} \frac{\partial^2 G}{\partial x_j \partial x_k} - \frac{\lambda}{2(\lambda + \mu)} \frac{\partial^2 F}{\partial x_j^2} \frac{\partial^2 G}{\partial x_k^2}\right) dx \tag{6.10}$$

We note that the functions in (6.6) are equal to zero at the origin of coordinates. From this and formulae (6.7), we conclude that the principal term (6.5) of the asymptotic form of the Airy function on a ligament has the form $A_0(3t^2 - 2t^3)$. According to (6.10) and (6.8), (6.6), the value of A_0 depends globally on the load p (that is, on its values at all points of Γ) and to find A_0 we need to know the function w_0 completely. Thus, as in the solution of (1.3), (1.4), we obtain integral formulae of the type (3.9).

7. THREE-DIMENSIONAL LIGAMENTS

Suppose that the three-dimensional elastic body Ω_ϵ has a hole, the boundary Γ_ϵ of which has approached the outer surface Γ of the body. We denote by $\omega \subset \Gamma$ the set to which surfaces Γ and Γ_ϵ stick as $\epsilon \rightarrow 0$. Thus, Ω_ϵ contains a thin ligament in the neighbourhood of ω. The algorithm for constructing the asymptotic form for the problem of the theory of elasticity in Ω depends on the structure of the set ω. If ω is a smooth closed contour, this algorithm does not differ essentially from that given in Secs 2–4; the only new factor is the dependence of the limit problem (2.7), (2.8) on the variable s on the contour $\partial\omega$. In the case where ω consists of the two-dimensional region on Γ and its smooth boundary $\partial\omega$, the methods described in [20, 16] can be used to construct the asymptotic form of the stress–deformation state of the body (a plane region with a hole separated from the outer contour by a thin "beam" has been investigated in the same way in [4, 5, 16]).

We will consider the case in which the set ω consists of the point O only. Near O, Ω is defined by the relations

$$\begin{gathered} -h_-(x') \leqslant x_3 \leqslant \epsilon + h_+(x') \\ h_\pm(x') = r^{2m}(a_\pm(\varphi) + O(r)), \quad r \rightarrow 0 \end{gathered} \tag{7.1}$$

Here $x' = (x_1, x_2)$ are Cartesian coordinates in the plane, (r, φ) are the corresponding polar coordinates; $a_\pm(\varphi)$ are smooth functions on the circumference; $a_+ + a_- > 0$; $m = 1, 2, \ldots$.

In Ω_ϵ we will examine problem (1.3), (1.4) in which $p = (p_1, p_2, p_3)$ is a smooth self-balanced load on Γ, and, for simplicity, we take $p = 0$ near 0. As in the two-dimensional case, we will look for the asymptotic form of the solution by considering two limiting problems. The first is Eq. (1.3) in Ω_0 with boundary conditions (1.4) on $(\Gamma \cup \Gamma_0)\backslash 0$. The second is a system of equations in the $\mathbf{R}^2$ plane, which describes the effect of a boundary layer on the ligament. This system is constructed in the same way as in Sec. 2. Below we will refer to the formulae of Sec. 2 with variables

$$\xi = (\xi', \xi_3), \quad \xi' = (\xi_1, \xi_2) = \epsilon^{-\gamma} x', \quad \xi_3 = \epsilon^{-1} x_3, \quad \gamma = (2m)^{-1} \tag{7.2}$$

instead of those in (1.5).

In the asymptotic series (2.4) $u^j(\xi')$ and $U^j(\xi)$ are three-dimensional vectors $u^j = (v^{j\prime}, w^j)$, $v^{j\prime} = (v_1^j, v_2^j)$. Let $v^{1\prime} \equiv v$, $w^0 \equiv w$, and suppose that the other variables $v^{j\prime}$, w^j are equal to zero. Splitting the operators as in (2.1) and (2.3), we obtain the representation [see (2.5)]:

$$\begin{aligned} &U^{0\prime} = 0, \quad U_3^0 = 0, \quad U^{1\prime} = -\xi_3 \nabla w, \quad U_3^1 = 0 \\ &U^{2\prime} = 0, \quad U_3^2 = \lambda(\lambda + 2\mu)^{-1}(\tfrac{1}{2}\xi_3^2 \Delta w - \xi_3 \nabla \cdot v) \\ &U^{3\prime} = (\lambda + 2\mu)^{-1}(3\lambda + 4\mu)\nabla \Delta w \tfrac{1}{6}\xi_3^3 - [\nabla v + 2(\lambda + 2\mu)^{-1}(\lambda + \mu)\nabla \nabla \cdot v]\tfrac{1}{2}\xi_3^2 + \\ &+ (2\mu)^{-1}[Q(H_+ - H_-)v - \tfrac{1}{2}Q(H_+^2 + H_-^2)\nabla w] \end{aligned} \tag{7.3}$$

$$H_+(\epsilon, \xi') = 1 + a_+(\varphi)\rho^{2m}, \quad H_-(\epsilon, \xi') = a_-(\varphi)\rho^{2m}; \quad \rho = |\xi'| \tag{7.4}$$

$$\begin{aligned} &Q = (Q_1, Q_2), \quad Q_j(X; \xi', \nabla)v(\xi') = \sum_{k=1}^{2} \frac{\partial}{\partial \xi_k}\{X(\xi')[\mu(\frac{\partial v_k}{\partial \xi_j}(\xi') + \frac{\partial v_j}{\partial \xi_k}(\xi')) + \\ &+ \frac{2\lambda\mu}{\lambda + 2\mu}\delta_{j,k}\nabla \cdot v(\xi')]\} \end{aligned}$$

The solvability conditions for the calculation of $U^{3\prime}$ and U_2^4 form a system of partial differential equations

$$-Q(H_+ + H_-; \xi', \nabla)v + \tfrac{1}{2}Q(H_+^2 - H_-^2; \xi', \nabla)\nabla w = F' \tag{7.5}$$

$$\nabla \cdot \{-\tfrac{1}{2}Q(H_+^2 - H_-^2; \xi', \nabla)v + \tfrac{1}{3}Q(H_+^3 + H_-^3; \xi', \nabla)\nabla w\} = F_2 \tag{7.6}$$

Near the point O, the asymptotic form of the solution of the limiting problem is Ω_0 can also be described by exponential solutions of the system (7.5), (7.6), where $\xi' = x$ and $H_\pm(x') = r^{2m}a_\pm(\varphi)$

(see [21, 22]). A complication arising when investigating system (7.4), (7.5) is that the differential operators that it contains degenerate at $r = 0$ (or increase inconsistently as $\rho \to +\infty$). However, by multiplying Eqs (7.5) and (7.6) by r^{-2m} and r^{-4m}, respectively, and replacing the function w by $\mathbf{w} = r^{2m} w$, the system is brought to a form which can be used with the results of [23, 24]. Thus, the exponential solutions mentioned have the form

$$v(x') = r^{\Lambda} \sum_{j=0}^{k} (j!)^{-1} (\ln r)^j V^{(k-j)}(\varphi)$$
$$w(x') = r^{\Lambda+1-2m} \sum_{j=0}^{k} (j!)^{-1} (\ln r)^j W^{(k-j)}(\varphi) \tag{7.7}$$

Here $k = 0, \ldots, J-1$, and Λ and $(V^{(0)}, W^{(0)}), \ldots, (V^{(J-1)}, W^{(J-1)})$ are the eigenvalue and Jordan chain (eigenvectors and adjoint vectors) of a bundle constructed in a standard way [23], on the unit circle.

Six solutions can be written explicitly.

$$(1,0,0), (0,1,0), (0,0,1), (-x_2, x_1, 0), (0,0,x_1), (0,0,x_2)$$

These correspond to eigenvalues $\Lambda = 0$, $2m-1$, 1, $2m$ and, by virtue of (7.3), generate rigid displacements which also determine the principal term of the asymptotic form of the solution u^0 of the three-dimensional problem in the limit region Ω_0

$$u_1^0(x) = c_1 - b_3 x_2 + b_2 x_3 + O(r^{\operatorname{Re}\Lambda_0})$$
$$u_2^0(x) = c_2 + b_3 x_1 + b_1 x_3 + O(r^{\operatorname{Re}\Lambda_0}) \tag{7.8}$$
$$u_3^0(x) = c_3 - b_2 x_1 - b_1 x_2 + O(r^{\operatorname{Re}\Lambda_0 + 1 - 2m})$$

Here Λ_0 is the first eigenvalue (with smallest real part) of the bundle in the half-plane $\{\Lambda \in \mathbf{C}: \operatorname{Re}\Lambda > -m\}$ which is different from the numbers $0, 1, 2m-1, 2m$.

We should emphasize that (7.8) is very different from (3.1): in the two-dimensional case, the remainder decays exponentially as one approaches the top of the peak, but exhibits exponential behaviour in the three-dimensional case only as $r \to 0$. This complicates the procedure of matching the total, outer and inner expansions, although the principal term of the boundary layer is easier to find in this problem. The point is that there is another difference between (7.8) and (3.1) due to the different geometries of the limit regions. In the problem of Sec. 3, the point was the top of two peaks at once, the series (3.1) for the two peaks containing different constants, $k_j^{\pm}$, and solutions (2.15) which allow for discontinuities $k_j^+ - k_j^-$ are included in the two-dimensional boundary layer (other possibilities associated with the geometry of the set Ω_0 were discussed in point 4 of Sec. 5). The series (7.8) has the same form, however the point O is approached, there are no discontinuities, and the principal terms of the boundary layer are given by the formula

$$(c_1, c_2, c_3) + \epsilon^{\gamma}(-b_3\xi_2, b_3\xi_1, -b_2\xi_1 - b_1\xi_2) + \epsilon(b_2\xi_3, b_1\xi_3, 0) \tag{7.9}$$

Note that (7.9) corresponds to zero stresses, and so the asymptotic form of the stress–deformation state of the ligament is determined by the lowest terms of the boundary layer. To construct these, we need to find special solutions (7.7) of the uniform system (7.5), (7.6) in the case when $H_{\pm}(x') = r^{2m} a_{\pm}(\varphi)$, calculate the coefficients of those solutions in the expansion of the field u^0 and solve system (7.5), (7.6) in the case (7.3), which has the given asymptotic form at infinity [see formulae (3.1), (3.2), (3.9) and (4.1), (2.15), for which it was possible to find explicit solutions, owing to the fact that (2.7), (2.8) is a system of ordinary differential equations].

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